Maths Class 11 Chapter 5 Part -1 Quadratic equations

1. **Real Polynomial:** Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is called a real polynomial of real variable x with real coefficients.

2. **Complex Polynomial:** If $a_0, a_1, a_2, \ldots, a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.

3. **Degree of a Polynomial:** A polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$, real or complex is a polynomial of degree n , if $a_n \neq 0$.

4. **Polynomial Equation:** If f(x) is a polynomial, real or complex, then f(x) = 0 is called a polynomial equation. If f(x) is a polynomial of second degree, then f(x) = 0 is called a quadratic equation .

Quadratic Equation: A polynomial of second degree is called a quadratic polynomial. Polynomials of degree three and four are known as cubic and biquadratic polynomials respectively. A quadratic polynomial f(x) when equated to zero is called quadratic equation. i.e., $ax^2 + bx + c = 0$ where $a \neq 0$.

Roots of a Quadratic Equation: The values of variable x .which satisfy the quadratic equation is called roots of quadratic equation.

Important Points to be Remembered

- An equation of degree n has n roots, real or imaginary .
- Surd and imaginary roots always occur in pairs of a polynomial equation with real coefficients i.e., if $(\sqrt{2} + \sqrt{3}i)$ is a root of an equation, then' $(\sqrt{2} - \sqrt{3}i)$ is also its root. .
- An odd degree equation has at least one real root whose sign is opposite to that of its last' term (constant term), provided that the coefficient of highest degree term is positive.
- Every equation of an even degree whose constant term is negative and the coefficient of highest degree term is positive has at least two real roots, one positive and one negative.
- If an equation has only one change of sign it has one positive root.
- If all the terms of an equation are positive and the equation involves odd powers of x, then all its roots are complex.

Solution of Quadratic Equation

1.**Factorization Method:** Let $ax^2 + bx + c = \alpha(x - \alpha)(x - \beta) = O$. Then, $x = \alpha$ and $x = \beta$ will satisfy the given equation.

2. **Direct Formula:** Quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots, given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha = \frac{-b+\sqrt{D}}{2a}, \beta = \frac{-b-\sqrt{D}}{2a}$$

where $D = \Delta = b^2 - 4ac$ is called discriminant of the equation .

Above formulas also known as Sridharacharya formula.

Nature of Roots

Let quadratic equation be $ax^2 + bx + c = 0$, whose discriminant is D.

(i) For $ax^2 + bx + c = 0$; a, b , $C \in R$ and $a \neq 0$, if

(a) D < => Complex roots

(b) $D > 0$ => Real and distinct roots

(c) $D = 0$ => Real and equal roots as $\alpha = \beta = -b/2a$

(ii) If a, b, $C \in Q$, $a \neq 0$, then

(a) If $D > 0$ and D is a perfect square => Roots are unequal and rational.

(b) If $D > 0$, $a = 1$; b, $c \in I$ and D is a perfect square. => Roots are integral. .

(c) If D > and D is not a perfect square. => Roots are irrational and unequal.

(iii) **Conjugate Roots** The irrational and complex roots of a quadratic equation always occur in pairs. Therefore,

(a) If one root be $\alpha + i\beta$, then other root will be $\alpha - i\beta$.

(b) If one root be $\alpha + \sqrt{\beta}$, then other root will be $\alpha - \sqrt{\beta}$.

(iv) If D, and D2 be the discriminants of two quadratic equations, then

(a) If $D_1 + D_2 \geq 0$, then At least one of D_1 and $D_2 \geq 0$ If $D_1 < 0$, then $D_2 > 0$,

(b) If $D_1 + D_2 < 0$, then At least one of D_1 and $D_2 < 0$ If $D_1 > 0$, then $D_2 < 0$

Roots Under Particular Conditions

For the quadratic equation $ax^2 + bx + e = 0$.

(i) If b = 0 => Roots are real/complex as (c < 0/c > 0) and equal in magnitude but of opposite sign.

(ii) If c = 0 => One roots is zero, other is – b / a.

(iii) If b = C = 0 => Both roots are zero.

(iv) If a = c => Roots are reciprocal to each other.

(v) If a > 0, c < 0, a < 0, c > 0} => Roots are of opposite sign.

(vi) If a > 0, b > 0, c > 0, a < 0, b < 0, c < 0} => Both roots are negative, provided D ≥ 0

(vii) If a > 0, b < 0, c > 0, a < 0, b > 0, c < 0} => Both roots are positive, provided D ≥ 0

(viii) If sign of a = sign of b ≠ sign of c => Greater root in magnitude is negative.

(ix) If sign of b = sign of c ≠. sign of a => Greater root in magnitude is positive.

(x) If a + b + c = 0 => One root is 1 and second root is c/a.

Relation between Roots and Coefficients

1. **Quadratic Equation:** If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are α and β, then Sum of roost = S = α + β = -b/a = – coefficient of x / coefficient of x^2 Product of roots = P = α * β = c/a = constant term / coefficient of x^2

2. **Cubic Equation:** If α, β and γ are the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$.

Then,

$$\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a}$$
$$\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$
$$\alpha\beta\gamma = -\frac{d}{a}$$

3. **Biquadratic Equation:** If α, β, γ and δ are the roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma = -\frac{b}{a},$$
$$S_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = (-1)^2\frac{c}{a} = \frac{c}{a}$$

$$S_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

$$S_3 = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a}$$

$$S_4 = \alpha \cdot \beta \cdot \gamma \cdot \delta = (-1)^4 \frac{e}{a} = \frac{e}{a}$$

Symmetric Roots: If roots of quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) are α and β, then

(i) $(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm \frac{\sqrt{b^2 - 4ac}}{a} = \frac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm \frac{b\sqrt{b^2 - 4ac}}{a^2} = \frac{\pm b\sqrt{D}}{a^2}$

(iv) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = -\frac{b(b^2 - 3ac)}{a^3}$

(v) $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta) = \frac{\pm(b^2 - ac)\sqrt{b^2 - 4ac}}{a^3}$

(vi) $\alpha^4 + \beta^4 = \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2 = \left(\frac{b^2 - 2ac}{a^2}\right)^2 - \frac{2c^2}{a^2}$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2) = \frac{\pm b(b^2 - 2ac)\sqrt{b^2 - 4ac}}{a^4}$

(viii) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta = \frac{b^2 - ac}{a^2}$

(ix) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{b^2 - 2ac}{ac}$

(x) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta) = -\frac{bc}{a^2}$

(xi) $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2} = \frac{b^2D + 2a^2c^2}{a^2c^2}$

Formation of Polynomial Equation from Given Roots

If a_1, a_2 a_3,…, an are the roots of an nth degree equation, then the equation is $x^n - S_1X^{n-1} + S_2X^{n-2} - S_3X^{n-3} + \ldots + (\_1)^n S_n = 0$ where S_n denotes the sum of the products of roots taken n at a time.

1. **Quadratic Equation**

If α and β are the roots of 'a quadratic equation, then the equation is $x^2 - S_1X + S_2 = 0$

i.e., $x^2 - (\alpha + \beta) x + \alpha\beta = 0$

2. **Cubic Equation**

If α, β and γ are the roots of cubic equation, then the equation is

$$x^3 - S_1x^2 + S_2x - S_3 = 0$$
$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$$

3. **Biquadratic Equation**

If α, β, γ and δ are the roots of a biquadratic equation, then the equation is

$$x^4 - S_1x^3 + S_2x^2 - S_3x + S_4 = 0$$
$$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \gamma\delta\alpha)x + \alpha\beta\gamma\delta = 0$$

Equation In Terms of the Roots of another Equation

If α, β are roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are.

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (replace x by $-x$)

(ii) $\alpha^n, \beta^n; n \in N \Rightarrow a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ (replace x by $x^{1/n}$)

(iii) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0$ (replace x by x/k)

(iv) $k + \alpha, k + \beta \Rightarrow a(x - k)^2 + b(x - k) + c = 0$ (replace x by $(x - k)$)

(v) $\frac{\alpha}{k}, \frac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$ (replace x by kx)

(vi) $\alpha^{1/n}, \beta^{1/n}; n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$ (replace x by x^n)

The quadratic function $f(x) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is always resolvable into linear factor, iff

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Condition for Common Roots in a Quadratic Equation

1. **Only One Root is Common**

If α be the common root of quadratic equations

$a_1x^2 + b_1x + C_1 = 0$,

and $a_2x^2 + b_2x + C_2 = 0$,

then $a_1a^2 + b_1\alpha + C_1 = 0$,

and $a_2a^2 + b_2\alpha + C_2 = 0$,

By Cramer’s Rule

$$\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}, \alpha \neq 0$$

Hence, the condition for only one root common is

$(c_1a_2 – c_2a_1)_2 = (b_1c_2 – b_2c_1)(a_1b_2 – a_2b_1)$

2. **Both Roots are Common**

The required condition is

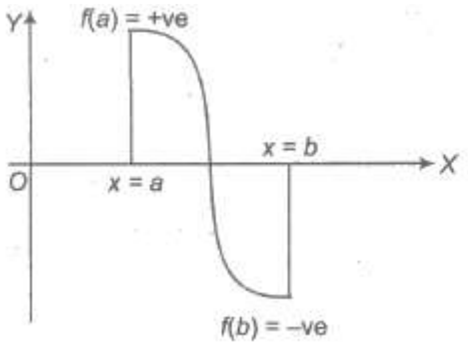
$a_1 / a_2 = b_1 / b_2 = c_1 / c_2$

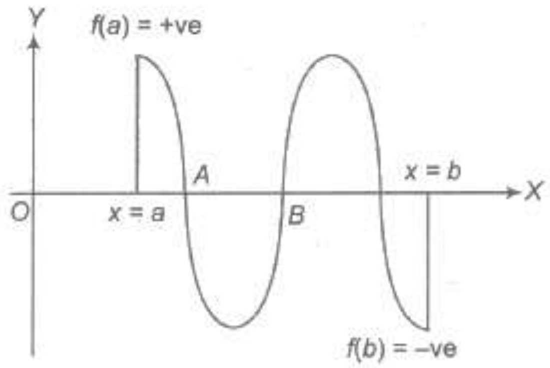
(i) To find the common root of two equations, make the coefficient of second degree term in the two equations equal and subtract. The value of x obtained is the required common root.

(ii) Two different quadratic equations with rational coefficient can not have single common root which is complex or irrational as imaginary and surd roots always occur in pair.

Properties of Quadratic Equation

(i) f(a) . f(b) < 0, then at least one or in general odd number of roots of the equation f(x) = 0 lies between a and b.





(ii) f(a) . f(b) > 0, then in general even number of roots of the equation f(x) = 0 lies between a and b or no root exist f(a) = f(b), then there exists a point c between a and b such that f'(c) = 0, a < c < b.

(iii) If the roots of the quadratic equation $a_1x^2 + b_1x + c_1 = 0$, $a_2x^2 + b_2x + c_2 = 0$ are in the ratio (i.e., &alpha$_1$;/β_1 = &alpha$_2$;/β_2), then

$b_1^2 / b_2^2 = a_1c_1 / a_2c_2$.

(iv) If one root is k times the other root of the quadratic equation $ax^2 + bx + c = 0$,then

$(k + 1)^2 / k = b^2 / ac$

Quadratic Expression

An expression of the form $ax^2 + bx + c$, where a, b, c $\in$ R and a $\neq$ 0 is called a quadratic expression in x .

1. **Graph of a Quadratic Expression**

We have

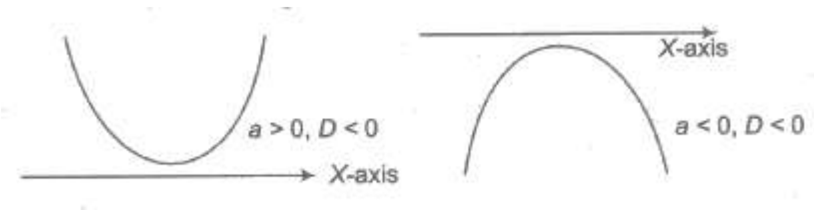
$y = ax^2 + bx + c = f(x)$

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right] \Rightarrow y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Let y + D/4a = Y and x + D / 2a = X

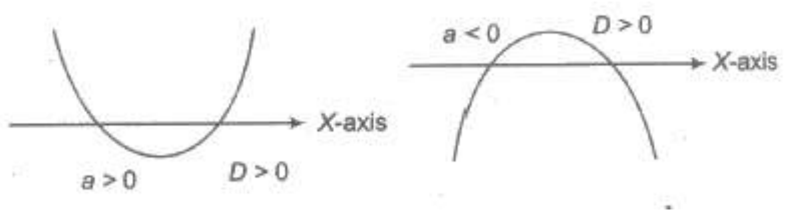
$Y = a * X^2 \Rightarrow X^2 = Y / a$

(i) The graph of the curve y = f(x) is parabolic.
(ii) The axis of parabola is X = 0 or x + b / 2a = 0 i.e., (parallel to Y-axis). •
(iii) If a > 0, then the parabola opens upward.
If a < 0, then the parabola opens downward.

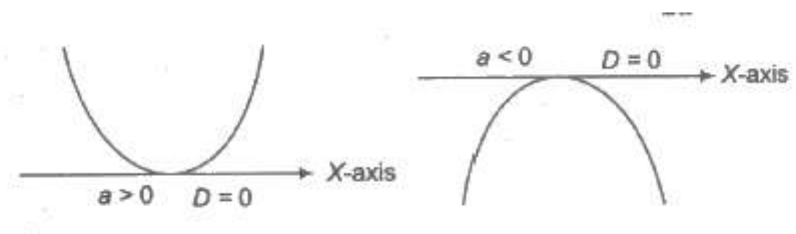


2. **Position of** $y = ax^2 + bx + c$ **with Respect to Axes.**

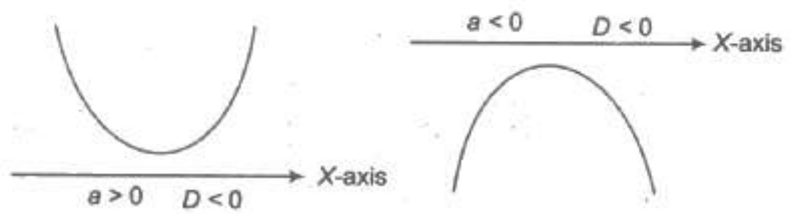
(i) For D > 0, parabola cuts X-axis in two real and distinct points
i.e, x = -b ± √D / 2a



(ii) For D = 0, parabola touch X-axis in one point, x = – b/2a.



(iii) For D < O,parabola does not cut X-axis (i.e., imaginary value of x).



3. **Maximum and Minimum Values of Quadratic Expression**

(i) If $a > 0$, quadratic expression has least value at $x = b / 2a$. This least value is given by $4ac – b^2 / 4a = – D/4a$. But their is no greatest value.

(ii) If $a < 0$, quadratic expression has greatest value at $x = – b/2a$. This greatest value is given by $4ac – b^2 / 4a = – D/4a$. But their is no least value.

4. **Sign of Quadratic Expression**

(i) $a > 0$ and $D < 0$, so $f(x) > 0$ for all $x \in R$ i.e., $f(x)$ is positive for all real values of x.

(ii) $a < 0$ and $D < 0$, so $f(x) < 0$ for all $x \in R$ i.e., $f(x)$ is negative for all real values of x.

(iii) $a > 0$ and $D = 0$, so $f(x) \geq 0$ for all $x \in R$ i.e., $f(x)$ is positive for all real values of x except at vertex, where $f(x) = 0$.

(iv) $a < 0$ and $D = 0$, so $f(x) \leq 0$ for all $x \in R$ i.e., $f(x)$ is negative for all real values of x except at vertex, where $f(x) = 0$.

(v) $a > 0$ and $D > 0$
Let $f(x) = o$ have two real roots α and β ($\alpha < \beta$), then $f(x) > 0$ for $x \in (-\infty, \alpha) \cup (\beta,\infty)$ and $f(x) < 0$ for all $x \in (\alpha, \beta)$.

(vi) $a < 0$ and $D > 0$
Let $f(x) = 0$ have two real roots α and β ($\alpha < \beta$). Then, $f(x) < 0$ for all $x \in (-\infty, \alpha) \cup (\beta,\infty)$ and $f(x) > 0$ for all
$x \in (\alpha, \beta)$. ,

5. **Intervals of Roots**

In some problems, we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c.
Since, $a \neq 0$, we can take $f(x) = x^2 + b/a\, x + c/a$.

(i) Both the roots are positive i.e., they lie in $(0,\infty)$, if and only if roots are real, the sum of the roots as well as the product of the roots is positive.

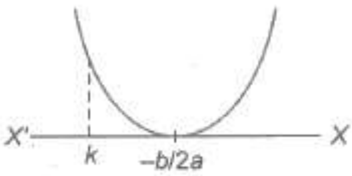
$\alpha + \beta = -b/a > 0$ and $\alpha\beta = c/a > 0$ with $b^2 – 4ac \geq 0$

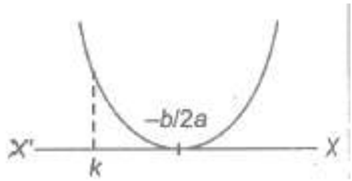
Similarly, both the roots are negative i.e., they lie in $(-\infty,0)$ ifF roots are real, the sum of the roots is negative and the product of the roots is positive.

i.e., $\alpha + \beta = -b/a < 0$ and $\alpha\beta = c/a > 0$ with $b^2 – 4ac \geq 0$

(ii) Both the roots are greater than a given number k, iFf the following conditions are satisfied

$D \geq 0$, $-b/2a > k$ and $f(k) > 0$



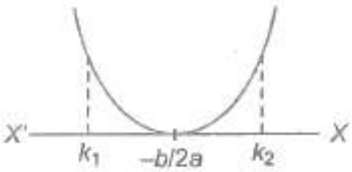


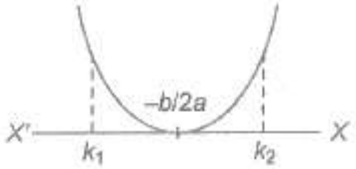
(iii) Both .the roots are less than a given number k, iff the following conditions are satisfied

$D \geq 0$, $-b/2a > k$ and $f(k) > 0$

(iv) Both the roots lie in a' given interval (k_1, k_2), iff the following conditions are satisfied

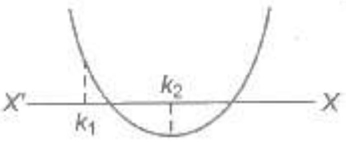
$D \geq 0, k_1 < -b/2a < k_2$ and $f(k_1) > 0$, $f(k_2) > 0$

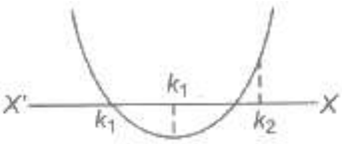




(v) Exactly one of the roots lie in a given interval (k_1, k_2), iff

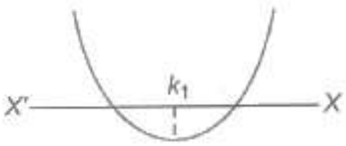
$f(k_1)\, f(k_2) < 0$





(vi) A given number k lies between the roots iff f(k) < O. In particular, the roots of the equation will be of opposite sign, iff 0 lies between the roots.

$\Rightarrow f(0) < 0$



Wavy Curve Method

Let $f(x) = (x - a_1)^{k_1} (x - a_2)^{k_2} (x - a_3)^{k_3} \ldots (x - a_{n-1})^{k_{n-1}} (x - a_n)^{k_n}$

where $k_1, k_2, k_3, \ldots, k_n \in N$ and $a_1, a_2, a_3, \ldots, a_n$ are fixed natural numbers satisfying the condition.

$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$..

First we mark the numbers $a_1, a_2, a_3, \ldots, a_n$ on the real axis and the plus sign in the interval of the right of the largest of these numbers, i.e., on the right of a_n. If k_n is even, we put plus sign on the left of $a_n$and if k_n is odd, then we put minus sign on the left of a_n In the next interval we put a sign according to the following rule.

When passing through the point a_{n-1} the polynomial f(x) changes sign . if k_{n-1} is an odd number and the polynomial f(x) has same sign if k_{n-1} is an even number. Then, we consider the next interval and put a sign in it using the same rule.

Thus, we consider all the intervals. The solution of $f(x) > 0$ is the union of all interval in which we have put the plus sign and the solution of $f(x) < 0$ is the union of all intervals in which we have put the minus Sign.

Descarte's Rule of Signs

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in f(x) .

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in f(x).

Rational Algebraic In equations

(i) **Values of Rational Expression P(x)/Q(x)** for Real Values of x, where **P(x) and Q(x) are Quadratic Expressions** To find the values attained by rational expression of the form $a_1x^2 + b_1x + c_1 / a_2x^2 + b_2x + c_2$

for real values of x.

(a) Equate the given rational expression to y.
(b) Obtain a quadratic equation in x by simplifying the expression,
(c) Obtain the discriminant of the quadratic equation.
(d) Put discriminant ≥ 0 and solve the in equation for y. The values of y so obtained determines the set of values attained by the given rational expression.

(ii) **Solution of Rational Algebraic In equation** If P(x) and Q(x) are polynomial in x, then the in equation $P(x) / Q(x) > 0$,
$P(x) / Q(x) < 0$, $P(x) / Q(x) \geq 0$ and $P(x) / Q(x) \leq 0$ are known as rational algebraic in equations.

To solve these in equations we use the sign method as

(a) Obtain P(x) and Q(x).
(b) Factorize P(x) and Q(x) into linear factors.
(c) Make the coefficient of x positive in all factors.
(d) Obtain critical points by equating all factors to zero.
(e) Plot the critical points on the number line. If these are n critical points, they divide the number line into (n + 1) regions.
(f) In the right most region the expression P(x) / Q(x) bears positive sign and in other region the expression bears positive and negative signs depending on the exponents of the factors .

Lagrange's identity

If $a_1, a_2, a_3, b_1, b_2, b_3 \neq R$, then

$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$

Algebraic Interpretation of Rolle's Theorem

Let f (x) be a polynomial having α and β as its roots such that $\alpha < \beta$, $f(\alpha) = f(\beta) = 0$.Also, a polynomial function is everywhere continuous and differentiable, then there exist $\theta \in (\alpha, \beta)$ such that $f'(\theta) = 0$. Algebraically, we can say between any two zeros of a polynomial f(x) there is always a derivative f' (x) = 0.

Equation and In equation Containing Absolute Value

1. **Equation Containing Absolute Value**

By definition, $|x| = x$, if $x \geq 0$ OR $-x$, if $x < 0$

If $|f(x) + g(x)| = |f(x)| + g(x)|$, then it is equivalent to the system $f(x) . g(x) \geq 0$.

If $|f(x) - g(x)| = |f(x)| - g(x)|$, then it is equivalent to the system $f(x) . g(x) \leq 0$.

2.**In equation Containing Absolute Value**

(i) $|x| < a \Rightarrow -a < x < a \ (a > 0)$
(ii) $|x| \leq a \Rightarrow -a \leq x \leq a$
(iii) $|x| > a \Rightarrow x < -a$ or $x > a$
(iv) $|x| \geq a \Rightarrow$ x le; – a or $x \geq a$

3. **Absolute Value of Real Number**

$|x| = -x, x < 0$ OR $+x, x \geq 0$

(i) $|xy| = |x||y|$
(ii) $|x / y| = |x| / |y|$
(iii) $|x|^2 = x^2$
(iv) $|x| \geq x$
(v) $|x + y| \leq |x| + |y|$
Equality hold when x and y same sign.
(vi) $|x - y| \geq ||x| - |y||$

Inequalities

Let a and b be real numbers. If a – b is negative, we say that a is less than b ($a < b$) and if a – b is positive, then a is greater than b ($a > b$).

Important Points to be Remembered

(i) If $a > b$ and $b > c$, then $a > c$. Generally, if $a_1 > a_2$, $a_2 > a_3$,…., $a_{n-1} > a_n$, then $a_1 > a_n$.

(ii) If $a > b$, then $a \pm c > b \pm c, \forall c \in R$

(iii) (a) If $a > b$ and $m > 0$, $am > bm, \frac{a}{m} > \frac{b}{m}$

(b) If $a > b$ and $m < 0$, $bm < am, \frac{b}{m} < \frac{a}{m}$

(iv) If $a > b > 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} < \frac{1}{b}$

(v) If $a < b < 0$, then

(a) $a^2 > b^2$ (b) $|a| > |b|$ (c) $\frac{1}{a} > \frac{1}{b}$

(vi) If $a < 0 < b$, then

(a) $a^2 > b^2$, if $|a| > |b|$

(b) $a^2 < b^2$, if $|a| < |b|$

(vii) If $a < x < b$ and a, b are positive real numbers then $a^2 < x^2 < b^2$

(viii) If $a < x < b$ and a is negative number and b is positive number, then

(a) $0 \le x^2 < b^2$, if $|b| > |a|$

(b) $0 \le x^2 \le b^2$, if $|a| > |b|$

(ix) If $\frac{a}{b} > 0$, then

(a) $a > 0$, if $b > 0$

(b) $a < 0$, if $b < 0$

(x) If $a_i > b_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$a_1 a_2 a_3 \ldots a_n > b_1 b_2 b_3 \ldots b_n$$

(xi) If $|x| < a$ and

(a) if a is positive, then $-a < x < a$.

(b) if a is negative, then $x \in \phi$

(xii) If $a_i > b_i$, where $i = 1, 2, 3, \ldots, n$, then

$a_1 + a_2 + a_3 + \ldots + a_n > b_1 + b_2 + \ldots + b_n$

(xiii) If $0 < a < 1$ and n is a positive rational number, then

(a) $0 < a^n < 1$ (b) $a^{-n} > 1$

Important Inequality

1. **Arithmetico-Geometric and Harmonic Mean Inequality**

(i) If a, b > 0 and a ≠ b, then

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2}{(1/a)+(1/b)}$$

(ii) if $a_i > 0$, where i = 1,2,3,…,n, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{1/n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

(iii) If $a_1, a_2, \ldots, a_n$ are n positive real numbers and $m_1, m_2, \ldots, m_n$ are n positive rational numbers, then

$$\frac{m_1 a_1 + m_2 a_2 + \ldots + m_n a_n}{m_1 + m_2 + \ldots + m_n} > \left(a_1^{m_1} \cdot a_2^{m_2} \cdot \ldots \cdot a_n^{m_n}\right)^{\frac{1}{m_1 + m_2 + \ldots + m_n}}$$

i.e., Weighted AM > Weighted GM

(iv) If $a_1, a_2, \ldots, a_n$ are n positive distinct real numbers, then

(a) $$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m \text{ if } m < 0 \text{ or } m > 1$$

(b) $$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m, \text{ if } 0 < m < 1$$

(c) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are rational numbers and M is a rational number, then

$$\frac{b_1a_1^m + b_2a_2^m + \ldots + b_na_n^m}{b_1 + b_2 + \ldots + b_n} > \left(\frac{b_1a_1 + b_2a_2 + \ldots + b_na_n}{b_1 + b_2 + \ldots + b_n}\right)^m, \text{if } 0 < m < 1$$

(d) $$\frac{b_1a_1^m + b_2a_2^m + \ldots + b_na_n^m}{b_1 + b_2 + \ldots + b_n} < \left(\frac{b_1a_1 + b_2a_2 + \ldots + b_na_n}{b_1 + b_2 + \ldots + b_n}\right)^m, \quad \text{if } 0 < m < 1$$

(v) If $a_1, a_2, a_3, \ldots, a_n$ are distinct positive real numbers and p, ,q, r are natural numbers, then

$$\frac{a_1^{p+q+r} + a_2^{p+q+r} + \ldots + a_n^{p+q+r}}{n} > \left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)\left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right)\left(\frac{a_1^r + a_2^r + \ldots + a_n^r}{n}\right)$$

2. **Cauchy – Schwartz's inequality**

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots, a_n^2) * (b_1^2 + b_2^2 + \ldots, b_n^2)$

Equality holds, iff $a_1 / b_1 = a_2 / b_2 = a_n / b_n$

3. **Tchebychef's Inequality**

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, such that

(i) If $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq b_3 \leq \ldots \leq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \geq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

(ii) If If $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq \ldots \geq b_n$, then

$n(a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n) \leq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

4. **Weierstrass Inequality**

(i) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then for $n \geq 2$

$(1 + a_1)(1 + a_2) \ldots (1 + a_n) > 1 + a_1 + a_2 + \ldots + a_n$

(ii) If $a_1, a_2, \ldots, a_n$ are real positive numbers, then

$(1 - a_1)(1 - a_2) \ldots (1 - a_n) > 1 - a_1 - a_2 - \ldots - a_n$

5. **Logarithm Inequality**

(i) (a) When $y > 1$ and $\log_y x > z \Rightarrow x > y^z$

(b) When $y > 1$ and $\log_y x < z \Rightarrow 0 < x < y^z$

(ii) (a) When $0 < y < 1$ and $\log_y x > z \Rightarrow 0 < x < y^z$

(b) hen $0 < y < 1$ and $\log_y x < z \Rightarrow x > y^z$

Application of Inequalities to Find the Greatest and Least Values

(i) If $x_l, x_2, \ldots, x_n$ are n positive variables such that $x_l + x_2 + \ldots + x_n = c$ (constant), then the product $x_l * x_2 * \ldots * x_n$ is greatest when $x_1 = x_2 = \ldots = x_n = c/n$ and the greatest value is $(c/n)^n$.

(ii) If $x_l, x_2, \ldots, x_n$ are positive variables such that $x_l, x_2, \ldots, x_n = c$ (constant), then the sum $x_l + x_2 + \ldots + x_n$ is least when $x_1 = x_2 = \ldots = x_n = c^{1/n}$ and the least value of the sum is $n(c^{1/n})$.

(iii) If $x_l, x_2, \ldots, x_n$ are variables and $m_l, m_2, \ldots, m_n$ are positive real number such that $x_l + x_2 + \ldots + x_n = c$ (constant), then $x_l^{m}{}_l * x_2^{m}{}_2 * \ldots * x_n^{m}{}_n$ is greatest, when

$x_l / m_l = x_2 / m_2 = \ldots = x_n / m_n$

$= x_l + x_2 + \ldots + x_n / m_l + m_2 + \ldots + m_n$

Maths Class 11 Chapter 5 Part -1 Complex Numbers

Imaginary Quantity

The square root of a negative real number is called an imaginary quantity or imaginary number. e.g., √-3, √-7/2

The quantity √-1 is an imaginary number, denoted by 'i', called iota.

Integral Powers of Iota (i)

$i=\sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4=1$

So, $i^{4n+1}= i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $i^{4n+4} = i^{4n} = 1$

In other words,

$i^n = (-1)^{n/2}$, if n is an even integer
$i^n = (-1)^{(n-1)/2}.i$, if is an odd integer

Complex Number

A number of the form z = x + iy, where x, y ∈ R, is called a complex number

The numbers x and y are called respectively real and imaginary parts of complex number z.

i.e., x = Re (z) and y = Im (z)

Purely Real and Purely Imaginary Complex Number

A complex number z is a purely real if its imaginary part is 0.

i.e., Im (z) = 0. And purely imaginary if its real part is 0 i.e., Re (z)= 0.

Equality of Complex Numbers

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, if $a_2= a_2$ and $b_1 = b_2$ i.e., Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2).

Algebra of Complex Numbers

1. Addition of Complex Numbers

Let $z_1 = (x_1 + iy_i)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their sum defined as

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Properties of Addition

(i) Commutative $z_1 + z_2 = z_2 + z_1$

(ii) Associative $(z_1 + z_2) + z_3 = + (z_2 + z_3)$

(iii) Additive Identity $z + 0 = z = 0 + z$

Here, 0 is additive identity.

2. Subtraction of Complex Numbers

Let $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their difference is defined as

$z_1 – z_2 = (x_1 + iy_1) – (x_2 + iy_2)$
$= (x_1 – x_2) + i(y_1 – y_2)$

3. Multiplication of Complex Numbers

Let $z_1 = (x_1 + iy_i)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their multiplication is defined as

$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 – y_1y_2) + i(x_1y_2 + x_2y_1)$

Properties of Multiplication

(i) Commutative $z_1z_2 = z_2z_1$

(ii) Associative $(z_1 z_2) z_3 = z_1(z_2 z_3)$

(iii) Multiplicative Identity $z • 1 = z = 1 • z$

Here, 1 is multiplicative identity of an element z.

(iv) Multiplicative Inverse Every non-zero complex number z there exists a complex number z_1 such that $z.z_1 = 1 = z_1 • z$

(v) Distributive Law

(a) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (left distribution)

(b) $(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$ (right distribution)

4. Division of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then their division is defined as

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$
$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

where z_2 # 0.

Conjugate of a Complex Number

If z = x + iy is a complex number, then conjugate of z is denoted by z

i.e., z = x – iy

Properties of Conjugate

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} \Leftrightarrow z$ is purely real

(iii) $z - \bar{z} \Leftrightarrow z$ is purely imaginary

(iv) $\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$

(v) $\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$

(vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(viii) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

(ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(x) $z_1 \bar{z}_2 \pm \bar{z}_1 z_2 = 2\operatorname{Re}(\bar{z}_1 z_2) = 2\operatorname{Re}(z_1 \bar{z}_2)$

(xi) $(\bar{z})^n = \overline{(z^n)}$

(xii) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

(xiii) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where $a_i, b_i, c_i; (i = 1, 2, 3)$ are complex numbers.

(xiv) $z\bar{z} = \{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2$

Modulus of a Complex Number

If z = x + iy, , then modulus or magnitude of z is denoted by |z| and is given by

$|z| = x^2 + y^2$.

It represents a distance of z from origin.

In the set of complex number C, the order relation is not defined i.e., $z_1 > z_2$ or $z_i < z_2$ has no meaning but $|z_1|>|z_2|$ or $|z_1| < |z_2|$ has got its meaning, since $|z|$ and $|z_2|$ are real numbers.

Properties of Modulus

(i) $|z| \geq 0$

(ii) If $|z| = 0$, then $z = 0$ i.e., $\text{Re}(z) = 0 = \text{Im}(z)$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z\bar{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1||z_2|$

In general

$|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3|\dots|z_n|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$

(viii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general

$$|z_1 \pm z_2 \pm z_3 \pm \dots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \dots + |z_n|$$

(ix) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

(x) $|z^n| = |z|^n$

(xi) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ greatest possible value of $|z_1 + z_2|$ is $|z_1| + |z_2|$ and least possible value of $|z_1 + z_2|$ is

$$||z_1| - |z_2||$$

(xii) $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\,\bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

(xiii) $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$$= |z_1|^2 + |z_2|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)$$

$$= |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\,\bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$$

(xiv) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(xv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xvi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xvii) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

where $a, b \in R$.

(xviii) z is **unimodulus**, if $|z| = 1$

Reciprocal/Multiplicative Inverse of a Complex Number

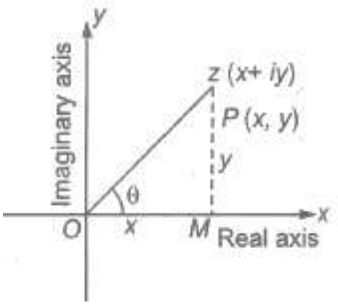
Let z = x + iy be a non-zero complex number, then

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$
$$= \frac{x-iy}{x^2+y^2}$$
$$= \frac{x}{x^2+y^2} + \frac{i(-y)}{x^2+y^2}$$

Here, z^{-1} is called multiplicative inverse of z.

Argument of a Complex Number

Any complex number z=x+iy can be represented geometrically by a point (x, y) in a plane, called Argand plane or Gaussian plane. The angle made by the line joining point z to the origin, with the x-axis is called argument of that complex number. It is denoted by the symbol arg (z) or amp (z).



Argument (z) = θ = $\tan^{-1}(y/x)$

Argument of z is not unique, general value of the argument of z is 2nπ + θ. But arg (0) is not defined.

A purely real number is represented by a point on x-axis.

A purely imaginary number is represented by a point on y-axis.

There exists a one-one correspondence between the points of the plane and the members of the set C of all complex numbers.

The length of the line segment OP is called the modulus of z and is denoted by |z|.

i.e., length of OP = $\sqrt{x^2 + y^2}$.

Principal Value of Argument

The value of the argument which lies in the interval $(-\pi, \pi]$ is called principal value of argument.

(i) If $x > 0$ and $y > 0$, then $\arg(z) = 0$
(ii) If $x < 0$ and $y > 0$, then $\arg(z) = \pi - 0$
(iii) If $x < 0$ and $y < 0$, then $\arg(z) = -(\pi - \theta)$
(iv) If $x > 0$ and $y < 0$, then $\arg(z) = -\theta$

Properties of Argument

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

In general,

$$\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots + \arg(z_n) + 2k\pi \ (k = 0, 1 \text{ or } -1)$$

(iii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi \ (k = 0, 1 \text{ or } = -1)$

(iv) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(v) $\arg\left(\dfrac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

(vi) $\arg(z^n) = n \arg(z) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

(vii) If $\arg\left(\dfrac{z_2}{z_1}\right) = \theta$, then $\arg\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta, k \in I$

(viii) If $\arg(z) = 0 \Rightarrow z$ is real

(ix) $\arg(z) - \arg(-z) = \begin{cases} \pi, & \text{if } \arg(z) > 0 \\ -\pi, & \text{if } \arg(z) < 0 \end{cases}$

(x) If $|z_1 + z_2| = |z_1 - z_2|$, then

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \dfrac{\pi}{2}$

(xi) If $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) = \arg(z_2)$

(xii) If $|z - 1| = |z + 1|$, then $\arg(z) = \pm\dfrac{\pi}{2}$

(xiii) If $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$, then $(z) = 1$

(xiv) If $\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{2}$, then z lies on circle of radius unity and centre at origin.

(xv) (a) If $z = 1 + \cos\theta + i\sin\theta$, then

$\arg(z) = \dfrac{\theta}{2}$ and $|z| = 2\cos\dfrac{\theta}{2}$

(b) If $z = 1 + \cos\theta - i\sin\theta$, then

$\arg(z) = -\dfrac{\theta}{2}$ and $|z| = 2\cos\dfrac{\theta}{2}$

(c) If $z = 1 - \cos\theta + i\sin\theta$, then

$\arg(z) = \dfrac{\pi}{2} - \dfrac{\theta}{2}$ and $|z| = 2\sin\dfrac{\theta}{2}$

(d) If $z = 1 - \cos\theta - i\sin\theta$, then

$\arg(z) = \dfrac{\pi}{4} - \dfrac{\theta}{2}$ and $|z| = \sqrt{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)$

(xvi) If $|z_1| \le 1, |z_2| \le 1$, then

(a) $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$

(b) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$

Square Root of a Complex Number

If z = x + iy, then

$$\sqrt{z} = \sqrt{x + iy} = \pm\left[\frac{\sqrt{|z| + x}}{2} + i\,\frac{\sqrt{|z| - x}}{2}\right], \text{ for } y > 0$$

$$= \pm\left[\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right], \text{ for } y < 0$$

Polar Form

If z = x + iy is a complex number, then z can be written as

z = |z| (cos θ + i sin θ) where, θ = arg (z)

this is called polar form.

If the general value of the argument is 0, then the polar form of z is

z = |z| [cos (2nπ + θ) + i sin (2nπ + θ)], where n is an integer.

Eulerian Form of a Complex Number

If z = x + iy is a complex number, then it can be written as

$z = re^{i0}$, where

r = |z| and θ = arg (z)

This is called Eulerian form and e^{i0}= cosθ + i sinθ and e^{-i0} = cosθ — i sinθ.

De-Moivre’s Theorem

A simplest formula for calculating powers of complex number known as De-Moivre’s theorem.

If n ∈ I (set of integers), then $(\cos\theta + i \sin\theta)^n$ = cos nθ + i sin nθ and if n ∈ Q (set of rational numbers), then cos nθ + i sin nθ is one of the values of $(\cos \theta + i \sin \theta)^n$.

(i) If $\frac{p}{q}$ is a rational number, then

$$(\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$$

(ii) $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta - i\sin\theta)^n$

(iii) More generally, for a complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\begin{aligned} z^n &= r^n(\cos\theta + i\sin\theta)^n \\ &= r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} \end{aligned}$$

(iv) $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)\right]$

(v) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

$$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

(vi) $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

(vii) $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

The nth Roots of Unity

The nth roots of unity, it means any complex number z, which satisfies the equation $z^n = 1$ or $z = (1)^{1/n}$

or $z = \cos(2k\pi/n) + i\sin(2k\pi/n)$, where k = 0, 1, 2, … , (n — 1)

Properties of nth Roots of Unity

1. nth roots of unity form a GP with common ratio $e^{(i2\pi/n)}$.
2. Sum of nth roots of unity is always 0.
3. Sum of nth powers of nth roots of unity is zero, if p is a multiple of n
4. Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.
5. Sum of pth powers of nth roots of unity is n, ifp is a multiple of n.
6. Product of nth roots of unity is $(-1)^{(n-1)}$.
7. The nth roots of unity lie on the unit circle |z| = 1 and divide its circumference into n equal parts.

The Cube Roots of Unity

Cube roots of unity are 1, ω, ω^2,

where $\omega = -1/2 + i\sqrt{3}/2 = e^{(i2\pi/3)}$ and $\omega^2 = (-1 - i\sqrt{3})/2$

$\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^2$

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^{2r} =$
0, if r is not a multiple of 3.
3, if r is,a multiple of 3.

(ii) $\omega^3 = \omega^{3r} = 1$

(iii) $\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^2$

(iv) Cube roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into 3 equal parts.

(v) It always forms an equilateral triangle.

(vi) Cube roots of – 1 are -1, $-\omega$, $-\omega^2$.

Important Identities

(i) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(ii) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(iii) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(iv) $x^2 - xy + y^2 = (x + \omega y)(x + y\omega^2)$

(v) $x^2 + y^2 = (x + iy)(x - iy)$

(vi) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(vii) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(viii) $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$

or $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$

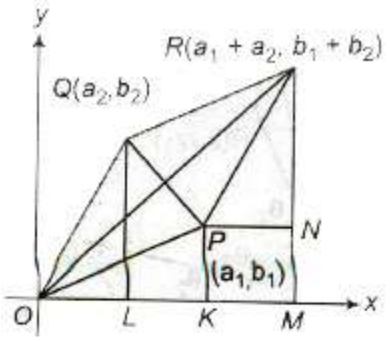
(ix) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Geometrical Representations of Complex Numbers

1. Geometrical Representation of Addition

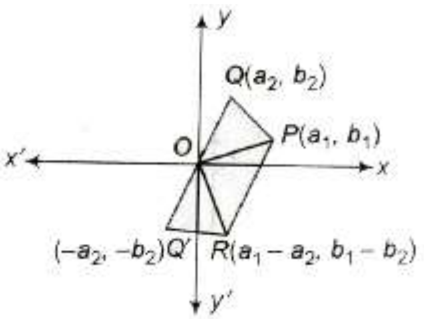
If two points P and Q represent complex numbers z_1 and z_2 respectively, in the Argand plane, then the sum $z_1 + z_2$ is represented

by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.



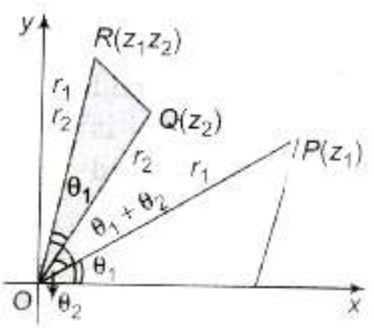
2. Geometrical Representation of Subtraction

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ia_2$ be two complex numbers represented by points P (a_1, b_1) and $Q(a_2, b_2)$ in the Argand plane. Q' represents the complex number ($—z_2$). Complete the parallelogram OPRQ' by taking OP and OQ' as two adjacent sides.



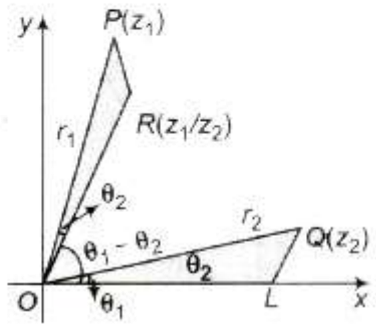
The sum of z_1 and $—z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ'. R represents the complex number $z_1 — z_2$.

3. Geometrical Representation of Multiplication of Complex Numbers



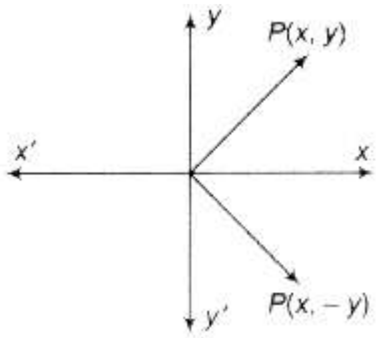
R has the polar coordinates $(r_1r_2, \theta_1 + \theta_2)$ and it represents the complex numbers z_1z_2.

4. Geometrical Representation of the Division of Complex Numbers



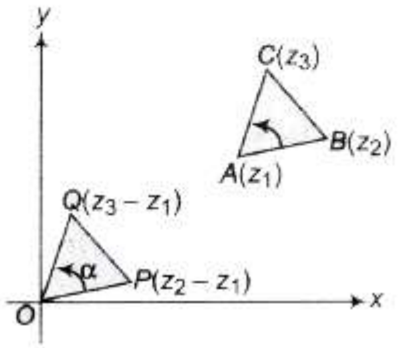
R has the polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$ and it represents the complex number z_1/z_2.
|z|=|z| and arg (z) = – arg (z). The general value of arg (z) is 2nπ – arg (z).

If a point P represents a complex number z, then its conjugate i is represented by the image of P in the real axis.

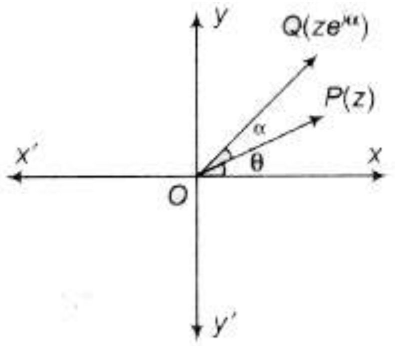


Concept of Rotation

Let z_1, z_2 and z_3 be the vertices of a ΔABC described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively. Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through angle a in anti-clockwise, sense it coincides with OQ.



Important Points to be Remembered



(a) $ze^{i\alpha}$ a is the complex number whose modulus is r and argument $\theta + \alpha$.
(b) Multiplication by $e^{-i\alpha}$ to z rotates the vector OP in clockwise sense through an angle α.

(ii) If z_1, z_2, z_3 and z_4 are the affixes of the points A, B,C and D, respectively in the Argand plane.

(a) AB is inclined to CD at the angle arg $[(z_2 - z_1)/(z_4 - z_3)]$.

(b) If CD is inclines at 90° to AB, then arg $[(z_2 - z_1)/(z_4 - z_3)] = \pm(\pi/2)$.

(c) If z_1 and z_2 are fixed complex numbers, then the locus of a point z satisfying arg $[([(z - z_1)/(z - z_2)] = \pm(\pi/2)$.

Logarithm of a Complex Number

Let z = x + iy be a complex number and in polar form of z is $re^{i\theta}$, then

$\log(x + iy) = \log(re^{i\theta}) = \log(r) + i\theta$

$\log(\sqrt{x^2 + y^2}) + i\tan^{-1}(y/x)$

or log(z) = log (|z|)+ iamp (z),

In general,

$z = re^{i(\theta + 2n\pi)}$

$\log z = \log|z| + i\arg z + 2n\pi i$

Applications of Complex Numbers in Coordinate Geometry

Distance between complex Points

(i) Distance between $A(z_1)$ and $B(_1)$ is given by

$AB = |z_2 — z_1| = \sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2}$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

(ii) The point P (z) which divides the join of segment AB in the ratio m : n is given by

$z = (mz_2 + nz_1)/(m + n)$

If P divides the line externally in the ratio m : n, then

$z = (mz_2 – nz_1)/(m – n)$

Triangle in Complex Plane

(i) Let ABC be a triangle with vertices A (z_1), B(z_2) and C(z_3) then

(a) Centroid of the ΔABC is given by

$z = 1/3(z_1 + z_2 + z_3)$

(b) Incentre of the AABC is given by

$z = (az_1 + bz_2 + cz_3)/(a + b + c)$

(ii) Area of the triangle with vertices A(z_1), B(z_2) and C(z_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

For an equilateral triangle,

$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$

(iii) The triangle whose vertices are the points represented by complex numbers z_1, z_2 and z_3 is equilateral, if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

i.e., $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Straight Line in Complex Plane

(i) The general equation of a straight line is az + az + b = 0, where a is a complex number and b is a real number.

(ii) The complex and real slopes of the line az + az are -a/a and – i[(a + a)/(a – a)].

(iii) The equation of straight line through z_1 and z_2 is $z = tz_1 + (1 — t)z_2$, where t is real.

(iv) If z_1 and z_2 are two fixed points, then $|z — z_1| = z — z_2|$ represents perpendicular bisector of the line segment joining z1 and z2.

(v) Three points z_1, z_2 and z_3 are collinear, if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

This is also, the equation of the line passing through $_1$, z_2 and z_3 and slope is defined to be $(z_1 – z_2)/z_1 – z_2$

(vi) Length of Perpendicular The length of perpendicular from a point z_1 to az + az + b = 0 is given by $|az_1 + az_1 + b|/2|a|$

(vii) arg $(z – z_1)/(z – z_2) = \beta$

Locus is the arc of a circle which the segment joining z_1 and z_2 as a chord.

(viii) The equation of a line parallel to the line az + az + b = 0 is az + az + λ = 0, where $\lambda \in R$.

(ix) The equation of a line parallel to the line az + az + b = 0 is az + az + iλ = 0, where $\lambda \in R$.

(x) If z_1 and z_2 are two fixed points, then I z — z11 =I z z21 represents perpendicular bisector of the segment joining A(z1) and B(z2).

(xi) The equation of a line perpendicular to the plane $z(z_1 – z_2) + z(z_1 – z_2) = |z_1|^2 – |z_2|^2$.

(xii) If z_1, z_2 and z_3 are the affixes of the points A, B and C in the Argand plane, then

(a) $\angle BAC = \arg[(z_3 – z_1/z_2 – z_1)]$

(b) $[(z_3 – z_1)/(z_2 – z_1)] = |z_3 – z_1|/|z_2 – z_1| (\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$.

(xiii) If z is a variable point in the argand plane such that arg (z) = θ, then locus of z is a straight line through the origin inclined at an angle θ with X-axis.

(xiv) If z is a variable point and z_1 is fixed point in the argand plane such that $(z — z_1)= \theta$, then locus of z is a straight line passing through the point z_1 and inclined at an angle θ with the X-axis.

(xv) If z is a variable point and z_1, z_2 are two fixed points in the Argand plane, then

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

Locus of z is the line segment joining z_1 and z_2.

(b) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

Locus of z is a straight line joining z_1 and z_2 but z does not lie between z1 and z_2.

(c) $\arg[(z - z_1)/(z - z_2)]$ = 0 or π

Locus z is a straight line passing through z_1 and z_2.

(d) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

Circle in Complete Plane

(i) An equation of the circle with centre at z_0 and radius r is

$|z - z_0| = r$

or $zz - z_0z - z_0z + z_0$

- $|z - z_0| < r$, represents interior of the circle.
- $|z - z_0| > r$, represents exterior of the circle.
- $|z - z_0| \leq r$ is the set of points lying inside and on the circle $|z - z_0| = r$. Similarly, $|z - z_0| \geq r$ is the set of points lying outside and on the circle $|z - z_0| = r$.
- **General equation of a circle is**

$zz - az - az + b = 0$

where a is a complex number and b is a real number. Centre of the circle = – a

Radius of the circle = $\sqrt{aa - b}$ or $\sqrt{|a|^2 - b}$

(a) Four points z_1, z_2, z_3 and z_4 are concyclic, if

$[(z_4 - z_1)(z_2 - z_3)]/[(z_4 - z_3)(z_2 - z_1)]$ is purely real.

(ii) $|z - z_1|/|z - z_2| = k \Rightarrow$ Circle, if $k \neq 1$ or Perpendicular bisector, if $k = 1$

(iii) The equation of a circle described on the line segment joining z_1 and $_1$ as diameter is $(z - z_1)(z - z_2) + (z - z_2)(z - z_1) = 0$

(iv) If z_1, and z_2 are the fixed complex numbers, then the locus of a point z satisfying arg $[(z - z_1)/(z - z_2)] = \pm \pi / 2$ is a circle having z_1 and z_2 at the end points of a diameter.

Conic in Complex plane

(i) Let z_1 and z2 be two fixed points, and k be a positive real number.

If $k > |z_1 - z_2|$, then $|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of the major axis is k.

(ii) Let z_1 and z2 be two fixed points and k be a positive real number.

If $k \neq |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = k$ represents hyperbola with foci at $A(z_1)$ and $B(z_2)$.

Important Points to be Remembered

- $\sqrt{-a} \times \sqrt{-b} \neq \sqrt{ab}$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is possible only, if both a and b are non-negative.

So, $i^2 = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

- is neither positive, zero nor negative.
- Argument of 0 is not defined.
- Argument of purely imaginary number is $\pi/2$
- Argument of purely real number is 0 or π.
- If $|z + 1/z| = a$ then the greatest value of $|z| = a + \sqrt{a^2} + 4/2$ and the least value of $|z| = -a + \sqrt{a^2} + 4/2$
- The value of $i^i = e^{-\pi 2}$
- The complex number do not possess the property of order, i.e., $x + iy < \text{(or)} > c + id$ is not defined.
- The area of the triangle on the Argand plane formed by the complex numbers z, iz and z + iz is $1/2|z|^2$.
- (x) If ω_1 and ω_2 are the complex slope of two lines on the Argand plane, then the lines are

(a) perpendicular, if $\omega_1 + \omega_2 = 0$.
(b) parallel, if $\omega_1 = \omega_2$.